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BROWNIAN MOVING AVERAGES HAVE CONDITIONAL FULL SUPPORT

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We prove that any Brownian moving average

$$X_t = \int_{-\infty}^{t} (f(s-t) - f(s))\, dB_s, \qquad t \geq 0,$$

satisfies the *conditional full support* condition introduced by Guasoni, Rásonyi and Schachermayer [*Ann. Appl. Probab.* **18** (2008) 491–520].

1. Introduction.

1.1. *Overview.* It is well known (see Soner, Shreve and Cvitanić [8], Levental and Skorokhod [6], Cherny [2]) that in the Black–Scholes–Merton model with proportional transaction costs the superreplication price of a European call option is equal to its trivial upper bound. The same is true for any European type contingent claim in this model (see Cvitanić, Pham and Touzi [3]). In the recent paper [4], Guasoni, Rásonyi and Schachermayer proved that the same result holds for a much wider class of models satisfying only a minor geometric condition termed *conditional full support* and denoted CFS for brevity (see the paper by Kabanov and Stricker [5] for further research in this direction).

The CFS condition is as follows. We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$ and a continuous $(\mathcal{F}_t)$-adapted process $(X_t)_{t\in[0,T]}$ meaning the discounted price (or the logarithm of the discounted price) of an asset. The CFS condition requires that, for any $t \in [0,T]$,

$$\operatorname{supp} \operatorname{Law}(X_u; t \leq u \leq T \mid \mathcal{F}_t) = C_{X_t}[t,T] \qquad \text{a.s.},$$

where $C_x[t,T]$ denotes the space of continuous real-valued functions on $[t,T]$ with $f(t) = x$ and "supp" denotes the support (the conditional distribution

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here is viewed as a measure on the space $C[t,T]$ of continuous functions on $[t,T]$).[1]

1.2. *Goal of the paper.* As motivated by the above discussion, the CFS condition is interesting and important. The paper [4] provides several examples of processes satisfying this condition. One of them is the fractional Brownian motion (FBM). It is well known (see Mandelbrot and Van Ness [7]) that FBM is a Brownian moving average, that is, it can be represented as

$$X_t = \int_{-\infty}^{t} (f(s-t) - f(s))\, dB_s, \qquad t \in [0,T], \tag{1.1}$$

with a certain function $f : \mathbb{R} \to \mathbb{R}$ such that $f = 0$ on $\mathbb{R}_+$ and $\int_{-\infty}^{t} (f(s-t) - f(s))^2\, ds < \infty$ for any $t \ge 0$. Let us remark that the class of moving averages includes processes that are, in a sense, more convenient for financial modeling than FBM; for example, FBM is not a semimartingale (except for two particular cases), while a moving average is a semimartingale provided that f is absolutely continuous and its derivative is square integrable on $(-\infty, 0]$ (see Cheridito [1]).

The main result of the paper is

THEOREM 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f = 0$ on $\mathbb{R}_+$, $\int_{-\infty}^{t} (f(s-t) - f(s))^2\, ds < \infty$ for any $t \ge 0$, and f is not zero on a set of positive Lebesgue measure. Then the process X defined by (1.1) satisfies the CFS condition with respect to its natural filtration.*

We also consider the CFS condition for general Gaussian processes. In discrete time it is easy to see that the CFS condition (appropriately redefined for the discrete-time case) is satisfied provided that X is a Gaussian process such that $\operatorname{Var}(X_t - X_s \mid X_u; u \le s) > 0$ for any $s < t$ (by Var we denote the variance). This might seem a bit surprising, but in continuous time the corresponding result does not hold; see Example 3.1.

2. Proof of Theorem 1.1. Let $T > 0$ and let $f \in L^2[-T, 0]$. For $g \in L^2[0,T]$, we denote by $f * g$ the convolution of f and g restricted to $[0,T]$, that is, the function

$$(f * g)(t) = \int_0^t f(s-t) g(s)\, ds, \qquad t \in [0,T].$$

[1]We deal with real-valued price processes, while Guasoni, Rásonyi and Schachermayer deal with strictly positive processes. The relationship between the two definitions is trivial: a process X satisfies our version of CFS if and only if e^X satisfies the CFS condition from [4].

LEMMA 2.1. *Let $h \in L^2[-T,0]$ satisfy the condition $\int_{-\varepsilon}^0 |h(t)|\,dt > 0$ for any $\varepsilon > 0$. Then the space $\{h * g : g \in L^2[0,T]\}$ is dense in $C_0[0,T]$.*

PROOF. If g is absolutely continuous with a square-integrable derivative and $g(0)=0$, then $(h*g)' = h*g'$. Thus, if a function $h*g$ approximates a function $\varphi \in L^2[0,T]$ in the L^2-sense, then the function $h*G$, where $G(t) = \int_0^t g(s)\,ds$, approximates the function $\Phi(t) = \int_0^t \varphi(s)\,ds$ in the $C_0[0,T]$-sense. So, it is sufficient to prove that the space $\{h*g : g \in L^2[0,T]\}$ is dense in $L^2[0,T]$.

Suppose that this is not true. Then there exists a function $\varphi \in L^2[0,T]$ not identically equal to zero and such that

$$\int_0^T (h*g)(t)\varphi(t)\,dt = 0 \qquad \forall g \in L^2[0,T].$$

This means that

$$\begin{aligned} 0 &= \int_0^T \int_0^t h(s-t)g(s)\varphi(t)\,ds\,dt \\ &= \int_0^T \int_s^T h(s-t)g(s)\varphi(t)\,dt\,ds \qquad \forall g \in L^2[0,T], \end{aligned}$$

which, in turn, is equivalent to the property

$$\int_s^T h(s-t)\varphi(t)\,dt = 0 \qquad \forall s \in [0,T].$$

But this is impossible due to the Titchmarsh convolution theorem (see [9], Chapter VI). The obtained contradiction yields the desired result. $\square$

PROOF OF THEOREM 1.1. Let $a \in (-\infty, 0]$ be a number such that $f = 0$ a.e. with respect to the Lebesgue measure on $[a,0]$ and $\int_{a-\varepsilon}^a |f(x)|\,dx > 0$ for any $\varepsilon > 0$. We can assume that $a = 0$. The case $a < 0$ is reduced to this one by considering the new Brownian motion $\widetilde{B}_t = B_{t-a} - B_{-a}$ and the new function $\widetilde{f}(x) = f(x-a)$.

We have to prove that, for any $t \in [0,T]$,

$$\operatorname{supp} \operatorname{Law}(X_u - X_t; t \le u \le T \mid \mathcal{F}_t) = C_0[t,T] \qquad \text{a.s.},$$

where $\mathcal{F}_t = \sigma(X_s; s \le t)$. Obviously, it is sufficient to prove the above property with $\mathcal{F}_t$ replaced by the larger filtration $\mathcal{G}_t = \sigma(B_s : -\infty < s \le t)$. With this substitution, it is obviously sufficient to check the property only for $t = 0$. We then have

$$\begin{aligned} &\operatorname{Law}(X_u; 0 \le u \le T \mid \mathcal{G}_0)(\omega) \\ &\quad = \operatorname{Law}\biggl(\int_0^u f(v-u)\,dB_v \end{aligned}$$

$$+\int_{-\infty}^{0}(f(v-u)-f(v))\,dB_v; 0\le u\le T\,\Big|\,\mathcal{G}_0\Big)(\omega)$$
$$=\operatorname{Law}\Big(\int_0^u f(v-u)\,dB_v+\varphi(u,\omega); 0\le u\le T\Big),$$

where $\varphi(\cdot,\omega)$ is the path of the process $Y=\int_{-\infty}^0(f(v-\cdot)-f(v))\,dB_v$ corresponding to the elementary outcome ω.

The above equality means that the conditional law of $(X_u)_{u\in[0,T]}$ given $\mathcal{G}_0$ is nothing but the unconditional law of $(\int_0^u f(v-u)\,dB_v)_{u\in[0,T]}$ shifted by the function $\varphi(u,\omega)$. As the two laws differ by such a shift, it is sufficient to prove that

$$\operatorname{supp}\operatorname{Law}\Big(\int_0^u f(v-u)\,dB_v; 0\le u\le T\Big)=C_0[0,T]. \tag{2.1}$$

It follows from the Girsanov theorem that, for any $g\in L^2[0,T]$,

$$\operatorname{Law}\Big(\int_0^u f(v-u)\,dB_v; u\le T\Big)$$
$$\sim\operatorname{Law}\Big(\int_0^u f(v-u)\,dB_v+\int_0^u f(v-u)g(v)\,dv; u\le T\Big).$$

Hence, if a function ψ belongs to the left-hand side of (2.1), then the same is true for $\psi+\int_0^\cdot f(v-\cdot)g(v)\,dv$. Using now the nonemptiness of the support and recalling Lemma 2.1, we obtain (2.1), which completes the proof. $\square$

3. Example. Let $(X_n)_{n=0,\dots,N}$ be a Gaussian random sequence such that

$$\operatorname{Var}(X_n-X_{n-1}\mid X_i; i\le n-1)>0\qquad\forall n=1,\dots,N. \tag{3.1}$$

Using induction in m, it is then easy to see that X satisfies the discrete-time version of the CFS condition:

$$\operatorname{supp}\operatorname{Law}(X_i : i=n+1,\dots,m\mid X_i : i=0,\dots,n)=\mathbb{R}^{m-n}\qquad\forall 0\le n<m\le N. \tag{3.2}$$

Let us remark that (3.2) obviously implies (3.1), so that the latter property serves as a criterion for the CFS for discrete-time Gaussian processes.

Surprisingly enough, in continuous time such a simple criterion does not hold, as shown by the next example.

EXAMPLE 3.1. Let B be a Brownian motion. For $n\in\mathbb{Z}_+$, denote $a_n=1-2^{-n}$ and let

$$X_t^n=b_n\int_0^t I(a_n\le s\le a_{n+1})\,dB_s$$
$$+\,b_n 2^{2n+3}\int_{a_n}^1(B_{s\wedge a_{n+1}}-B_{a_n})\,ds\int_0^t I(s\ge a_{n+1})\,ds,\qquad t\in[0,1].$$

The constants b_n are strictly positive and decrease to zero fast enough to ensure that

$$\sum_{n=0}^{\infty} \sup_{t\in[0,1]} |X_t^n| < \infty \qquad \text{a.s.}$$

Then the process

$$X_t = \sum_{n=0}^{\infty} X_t^n, \qquad t \in [0,1]$$

is continuous and Gaussian. For any $0 \le s < t \le 1$, the difference $X_t - X_s$ can be represented as $\xi_1 + \xi_2$, where ξ_1 is $\sigma(X_u; u \le s)$-measurable and ξ_2 is nondegenerate and depends on the increments of B after time s. Hence,

$$\operatorname{Var}(X_t - X_s \mid X_u; u \le s) > 0 \qquad \forall 0 \le s < t \le 1.$$

On the other hand,

$$\begin{aligned}\int_0^1 X_t\,dt &= \sum_{n=0}^{\infty} \int_0^1 X_t^n\,dt \\ &= \sum_{n=0}^{\infty} b_n \int_{a_n}^1 (B_{s\wedge a_{n+1}} - B_{a_n})\,ds \biggl[1 + 2^{2n+3} \int_{a_{n+1}}^1 (s - a_{n+1})\,ds\biggr] = 0,\end{aligned}$$

so that the CFS condition is violated for X already for $t = 0$.

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